

ASTRONOMY 9: HISTORY OF COSMOLOGY

Assignment #8—Solutions

2000 February 23

1. **How fast is a point at the equator moving due to the earth's rotation? Earth's radius is about 6400 km. Convert your answer to miles per hour, and compare to the speed of a typical airplane (500 mph).**

The speed at which something is moving is the distance traveled divided by the time taken. A point on Earth's equator moves a distance equal to the circumference of the earth, $C = 2\pi r \approx 40,000$ km, in one day, where $\pi = 3.14159265\dots$. So the speed is

$$\text{speed} = \frac{40,000 \text{ km}}{24 \text{ hours}} = 1670 \text{ kph} \times \left(\frac{1 \text{ mile}}{1.6 \text{ km}} \right) = 1040 \text{ mph.}$$

This is about twice as fast as the typical jet!

2. **Commander Ivanova lands on Z'ha'dum. Before her meeting with Zathras, she has a little time on her hands, so she decides to use Eratosthenes' method to measure the size of Z'ha'dum. At her landing point, she finds that the "sun" is directly overhead at noon and casts no shadow. She then walks 1000 km north (ok, she has a lot of time on her hands) and finds that her vertical 1-meter stick casts a shadow of length 32.5 centimeters (cm) at noon. What is the circumference of Z'ha'dum? What is its radius? Compare with the circumference of Earth (express your comparison as a ratio). Hint: Draw some pictures!**

From the attached figure, you can see that the angle we are after is

$$\theta = \tan^{-1} \frac{32.5 \text{ cm}}{100 \text{ cm}} = \tan^{-1} 0.325 \approx 18^\circ.$$

The function \tan^{-1} , sometimes called the arc-tangent or arctan, is the inverse of the tangent function. The ratio of the circumference to the distance between two points is clearly the same as the ratio of 360° to 18° :

$$\frac{C}{1000 \text{ km}} = \frac{360^\circ}{18^\circ} = 20,$$

which gives $C = 20,000$ km and a radius of $r = C/2\pi = 3200$ km. Z'ha'dum is therefore about one-half the size of the earth.

3. **How long would it take you to walk all the way around the equator of the Earth? Assume there is no water to get in the way, and that you can walk continuously at a brisk pace of 5 km per hour. Convert your answer to days.**

Again we know that speed is distance/time, which means that time = distance/speed. Using the circumference from #1,

$$t = \frac{40,000 \text{ km}}{5 \text{ km hr}^{-1}} = 8,000 \text{ hr} \times \left(\frac{1 \text{ day}}{24 \text{ hr}} \right) = 330 \text{ days.}$$

So it would take you about a year! Note the use of the -1 exponent with the hour unit; this is a shorthand for 1/hour, just like $10^{-1} = 1/10 = 0.1$. So km hr^{-1} is the same as km/hr or kph .

4. **Explain why Aristotle believed there is a "fifth element".**

Aristotle argued for the existence of a fifth element (in addition to air, fire, earth, and water) based on his ideas about motion. He divided the world into two realms where different physical laws applied: the imperfect sub-lunary (terrestrial) realm, and the perfect lunar (celestial) realm. In the terrestrial realm, natural motion was thought to be linear: air and fire moved up, water and earth down. But the heavenly motions appeared to be more circular than linear, and Aristotle believed that circular motion was the natural motion of the heavens. He thought that each basic natural motion corresponded to some fundamental element, and so he needed a fifth element, of which the heavens were made, for circular motion.

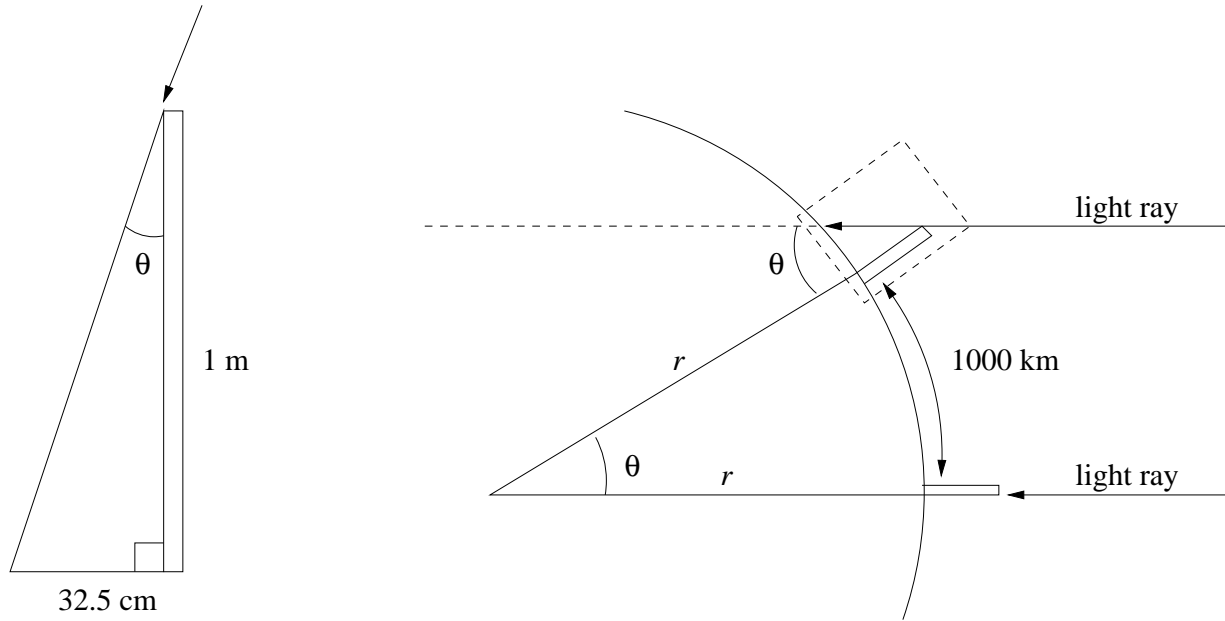


Figure 1: Eratosthenes' method for determining the circumference of a spherical planet with a distant sun.

5. What were Aristotle's reasons for believing in a spherical Earth? In what crucial way did his reasons differ from Plato's?

The reasons are given on p. 106 of the reader, in Aristotle's *On the Heavens*. Aristotle argues that the earth must be a sphere because the heavy material of which it is made is naturally attracted toward the center (today we recognize this as the gravitational attraction of matter; Aristotle thought heavy things just naturally moved toward the center of the universe). This naturally leads to a spherical shape.

Further reasons are based on observational evidence. First, he recognized that lunar eclipses are due to the shadow of the earth falling on the moon, and if the earth were any shape other than a sphere, you wouldn't always get the observed circular boundary of the shadow. Second, he knew traveling north or south "alters the circle of the horizon" so that you don't always see the same stars, which also implied a spherical (and small, compared to the distance to the stars) Earth.

On the other hand, Plato believed in a spherical Earth for largely *philosophical* reasons. He knew that the sphere was in a sense the most "perfect" (symmetric: looks the same from all directions, all points equally far from the center) solid. In a scientific sense, Aristotle's arguments are much more sophisticated. His first argument is based on *physical law*. Although he didn't really understand gravity and his physics was largely wrong, even the attempt to understand the cosmos in terms of physical law marked an important advance. The other arguments are based on *observations*, in which Plato was not really interested because he believed that the physical world was a mere imperfect shadow of the true underlying reality, the realm of Forms. Observing the natural world was therefore not a great priority for the Platonists.

6. In his work *The Sand Reckoner*, Archimedes (287–212 BC, famous for running naked through the streets after discovering the principle of buoyancy) computed that 10^{63} grains of sand would fill the known universe. Assume that each grain of sand has 10^{24} atoms, and each atom has a mass 1.6×10^{-27} kilograms (kg).

a) **How many atoms are there in all the sand?**

$$10^{63} \text{ grains} \times 10^{24} \text{ atoms/grain} = 10^{87} \text{ atoms.}$$

b) What is the mass of a grain of sand?

$$10^{24} \times 1.6 \times 10^{-27} \text{ kg} = 0.0016 \text{ kg}.$$

c) What is the total mass of all the sand?

$$0.0016 \text{ kg} \times 10^{63} = 1.6 \times 10^{60} \text{ kg}.$$

Note: the universe is really *much* bigger than this!

7. In class we discussed how Aristarchus estimated that the Sun is about 20 times farther away than the Moon. He did this by estimating that the Moon was half-full (“first quarter”) when it was separated from the Sun by an angle of 87° .

a) It turns out that he was wrong by *another* factor of 20; the Sun is really about 400 times farther than the Moon. Assuming that the Moon is in a uniform circular orbit (not a very good assumption for this problem, but never mind), what is the true angle between Moon and Sun at first quarter? Hint: Draw a picture!

From the figure, we see that

$$\theta = \cos^{-1} \frac{d_{\text{moon}}}{d_{\text{sun}}},$$

where d_{moon} is the distance to the moon, d_{sun} is the distance to the sun, and \cos^{-1} , sometimes called \arccos , is the inverse of the cosine function. If we let the ratio of distances be $1/400$ (close to the true value), then the angle is $\theta = 89.9^\circ$. Note how close this is to 90° !

Of course, the orbit of the moon is really an ellipse, not a circle, and the effect of this fact on the time of first-quarter cannot be neglected for this calculation.

Also, the calculation was much harder for Aristarchus, because the handy trigonometric functions (not to mention calculators) weren't around yet. He succeeded in placing the limit $18 < d_{\text{sun}}/d_{\text{moon}} < 20$.

b) By how much was Aristarchus in error about the angle? Express this error as a percentage.

We see that he was only off by about 2.9° , or

$$\frac{89.9^\circ - 87^\circ}{89.9^\circ} = 0.032 = 3.2\%.$$

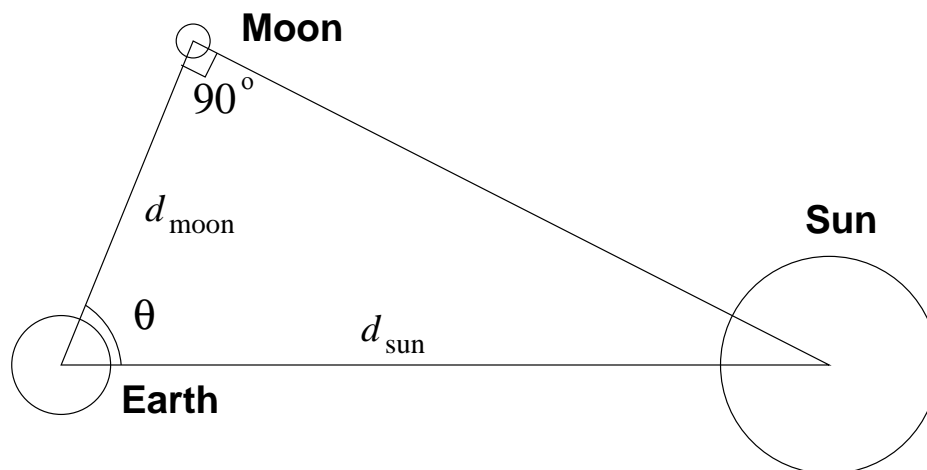


Figure 2: Aristarchus' method for determining the ratio $d_{\text{sun}}/d_{\text{moon}}$ from the angle θ between Moon and Sun at the time of first-quarter moon. Note: not to scale!

- c) **Is the error in the angle large compared to the error in the distance? What does this imply about the accuracy with which you would have to measure the angle in order to get an accurate measurement of the distance ratio? Does this seem like a good method?**

The error in the distance ratio was enormous—a factor of 20! But the error in the angle was much more modest, only 3.2%. To see what is happening, note that $d_{sun}/d_{moon} = \sec \theta$, where $\sec x = 1/\cos x$ is the secant function. Now look at what the secant does as the angle gets closer and closer to 90° . For example, $\sec 89^\circ = 57.3$, $\sec 89.9^\circ = 573$, $\sec 89.99^\circ = 5729$, In order to measure a large distance ratio, you need an extremely accurate measurement of the angle: an error of less than 0.1° could change the answer by more than a factor of 10! Judging exactly when first quarter occurs is not easy (especially since the moon is not a perfect sphere, but has mountains, craters, and valleys, although the Greeks did not know this). So the method, though very clever, is not a very practical way of getting accurate relative distances to the moon and sun. But it is easy to conclude that the sun is much farther away than the moon (because the angle is not much less than 90°) and must therefore be much larger, which may have motivated Aristarchus to place it at the center of the cosmos.